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On the existence of periodic solutions to p-Laplacian Rayleigh differential equation with a delay ☆

Shiping Lu [a,*], Zhanjie Gui [b]

[a] *Department of Mathematics, Anhui Normal University, Wuhu 241000, Anhui, PR China*
[b] *Department of Mathematics, Hainan Normal University, Haikou 571158, PR China*

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Abstract

In this paper, the authors study the existence of periodic solutions to a p-Laplacian Rayleigh differential equation with a delay as follows:

$$\big(\varphi_p\big(y'(t)\big)\big)' + f\big(y'(t)\big) + g\big(y\big(t-\tau(t)\big)\big) = e(t),$$

where $p > 1$ is a constant, $\varphi_p : R \to R$, $\varphi_p(u) = |u|^{p-2}u$, $f, g, e, \tau \in C(R, R)$, $\tau(t+T) \equiv \tau(t)$ with $\tau(t) \geqslant 0$, $\forall t \in [0, T]$, and $e(t+T) \equiv e(t)$, $T > 0$ is a constant. By using Mawhin's continuation theorem, some new results are obtained.

Keywords: p-Laplacian operator; Continuation theorem; Rayleigh differential equation

1. Introduction

The existence of periodic solutions to Raleigh differential equation with a deviating argument as follows:

$$x''(t) + f\big(x'(t)\big) + g\big(x\big(t-\tau(t)\big)\big) = e(t) \tag{1.1}$$

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* Corresponding author.
E-mail address: lushiping88@sohu.com (S. Lu).

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was studied in [1–3]. The growth conditions imposed on $f(x)$ are such as

$$\left|f(x)\right| \leqslant K, \quad \forall x \in R, \tag{1.2}$$

in [1],

$$xf(x) \geqslant \sigma|x|^{n+1} \quad \text{or} \quad xf(x) \leqslant -\sigma|x|^{n+1}, \quad \forall x \in R, \tag{1.3}$$

in [2] and

$$-K \leqslant f(x) \leqslant r_1|x| + K, \quad \forall x \in R, \tag{1.4}$$

in [3], where $K, n, \sigma > 0$, $r_1 \geqslant 0$ are constants. Meanwhile, the growth condition imposed on $g(x)$ is such as $|g(x)| \geqslant M$ for $x \leqslant -D$ in [1], which is named one-sided bounded type, and also the global linear growth condition

$$\lim_{|x|\to+\infty} \frac{g(x)}{x} = r \in [0, +\infty) \tag{1.5}$$

is required in [3]. In recent years, by using Mawhin's continuation theorem, Cheung and Ren [4] studied the existence of T-periodic solutions to p-Laplacian Liénard equation with a deviating argument as follows:

$$\big(\varphi_p\big(x'(t)\big)\big)' + f\big(x(t)\big)x'(t) + g\big(x\big(t-\tau(t)\big)\big) = e(t).$$

But the corresponding problem to p-Laplacian Rayleigh equation with a delay, as far as we know, was studied rarely. In this paper, we continue to study the existence of periodic solutions to a class of p-Laplacian Rayleigh differential equation with a delay as follows:

$$\big(\varphi_p\big(y'(t)\big)\big)' + f\big(y'(t)\big) + g\big(y\big(t-\tau(t)\big)\big) = e(t), \tag{1.6}$$

where $p > 1$ is a constant, $\varphi_p : R \to R$, $\varphi_p(u) = |u|^{p-2}u$, $f, g, e, \tau \in C(R, R)$, $\tau(t+T) \equiv \tau(t)$ with $\tau(t) \geqslant 0$, $\forall t \in [0, T]$, and $e(t+T) \equiv e(t)$ with $\int_0^T e(s)\,ds = 0$, $T > 0$ is a constant. By using Mawhin's continuation theorem, some new results are obtained. Clearly, Eq. (1.1) is the special case of Eq. (1.6) for $p = 2$. The significance of this paper is that even if for $p = 2$, the conditions imposed on $f(x)$ and $g(x)$ are different from the corresponding ones of [1–3]. For example, the conditions imposed on $g(x)$ and $f(x)$ are such as

$$\lim_{x\to-\infty} \frac{|g(x)|}{|x|^{p-1}} \leqslant r$$

and

$$\left|f(x)\right| \leqslant \alpha|x|^{p-1} + \beta, \quad \forall x \in R \text{ with } |x| > D,$$

where $r \geqslant 0$, $D > 0$, $\alpha \geqslant 0$ and $\beta > 0$ are constants. Also, we investigate the relation between the existence of periodic solutions and the delay $\tau(t)$. Furthermore, the methods to obtain a priori bounds of periodic solutions in [4] cannot be applied to present paper, since the crucial step $\int_0^T f(x(t))x'(t)\,dt = 0$ is no longer valid for Eq. (1.6).

2. Main lemmas

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. This means $X = \operatorname{Ker} L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$. Furthermore, let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ be the continuous projectors. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq 0$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to X$ is compact. Now, we recall this theorem in the first.

Lemma 2.1. [5] *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T-periodic solutions for Eq. (1.6), we should rewrite Eq. (1.6) in the following form:

$$\begin{cases} x_1'(t) = \varphi_q\big(x_2(t)\big) = \big|x_2(t)\big|^{q-2} x_2(t), \\ x_2'(t) = -f\big(\varphi_q\big(x_2(t)\big)\big) - g\big(x_1\big(t-\tau(t)\big)\big) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to Eqs. (2.1), then $x_1(t)$ must be a T-periodic solution to Eq. (1.6). Thus, in order to prove that Eq. (1.6) has a T-periodic solution, it suffices to show that Eqs. (2.1) has a T-periodic solution. Now, we set $C_T = \{\phi\colon\ \phi \in C(R,R),\ \phi(t+T) \equiv \phi(t)\}$ with the norm $|\phi|_0 = \max_{t\in[0,T]} |\phi(t)|$. $X = Y = \{x = (x_1(t), x_2(t)) \in C(R, R^2)\colon\ x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$. Clearly, X and Y are two Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \qquad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \tag{2.2}$$

$$N : X \to Y, \qquad Nx = \begin{pmatrix} \varphi_q(x_2) \\ -f(\varphi_q(x_2(t))) - g(x_1(t-\tau(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

It is easy to see that $\operatorname{Ker} L = R^2$, $\operatorname{Im} L = \{x\colon\ x \in Y,\ \int_0^T x(s)\,ds = 0\}$. So L is a Fredholm operator with index zero. Also let projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q$ defined by

$$Px = \frac{1}{T}\int_0^T x(s)\,ds; \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds,$$

and let K to represent the inverse of $L|_{\operatorname{Ker} P\cap D(L)}$. Obviously, $\operatorname{Ker} L = \operatorname{Im} Q = R^2$.

$$[Ky]^{-1}(t) = \int_0^T k(t,s)y(s)\,ds, \tag{2.4}$$

where

$$k(t,s) = \begin{cases} \frac{s}{T}, & 0 \leqslant s < t \leqslant T, \\ \frac{s-T}{T}, & 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

From (2.4), one can easily see that N is L-compact on $\overline{\Omega}$, where Ω is an open, bounded subset of X.

The following lemma is important for us to estimate a priori bounds of periodic solutions in Theorem 3.2.

Lemma 2.2. *Let $s \in C(R,R)$ with $s(t+T) \equiv s(t)$ and $s(t) \in [0,T]$, $\forall t \in R$. Suppose $\gamma = \max_{t\in[0,T]} s(t)$ and $u \in C^1(R,R)$ with $u(t+T) \equiv u(t)$. Then*

$$\int_0^T \left|u(t) - u\big(t - s(t)\big)\right|^p dt \leqslant \gamma^2 \int_0^T \left|u'(t)\right|^2 dt.$$

Lemma 2.2 is a direct consequence of the result obtained by Lu and Ge in [6].

For the sake of convince, we list the following assumptions which will be used for us to study the existence of T-periodic solutions to Eq. (1.6) in Section 3:

- [H_1] There is a constant $D > 0$ such that $g(x) < -|e|_0 - |f(0)|$ for $x > D$, and $g(x) > |e|_0 + |f(0)|$ for $x < -D$.
- [H_2] $\lim_{x\to-\infty} \frac{|g(x)|}{|x|^{p-1}} \leqslant r \in [0, +\infty)$.
- [H_3] There are constants $\alpha \geqslant 0$ and $\beta > 0$ such that $|f(y)| < \alpha|y|^{p-1} + \beta$, for $|y| > D$, where D is a constant defined by assumption [H_1].
- [H_4] Suppose $g \in C^1(R,R)$ and there is a constant $l > 0$ such that $-l \leqslant g'(x) \leqslant 0$, $\forall x \in R$.

3. Main results

Theorem 3.1. *Suppose assumptions* [H_1]–[H_3] *hold and $e(t)$ is not a constant. Furthermore, $\delta := \max_{t\in[0,T]} \tau(t) \in [0,T]$. Then Eq.* (1.6) *has at least one non-constant T-periodic solutions, if*

$$(2r+\alpha)\delta^{(p-1)/p}T^{1/p} + \alpha T < 1.$$

Proof. Considering the following operator equation:

$$Lx = \lambda Nx, \quad \lambda \in (0,1), \tag{3.1}$$

where L and N are defined by (2.2) and (2.3), respectively. Let $\Omega_1 = \{x\colon\ Lx = \lambda Nx,\ \lambda \in (0,1)\}$. If $x(t) = \binom{x_1(t)}{x_2(t)} \in \Omega_1$, then

$$\begin{cases} x_1'(t) = \lambda\varphi_q\big(x_2(t)\big) = \lambda\left|x_2(t)\right|^{q-2}x_2(t), \\ x_2'(t) = -\lambda f\big(\varphi_q\big(x_2(t)\big)\big) - \lambda g\big(x_1\big(t-\tau(t)\big)\big) + \lambda e(t). \end{cases} \tag{3.2}$$

Let t_0 be the maximum point of $x_1(t)$ on R, i.e., $x_1(t_0) = \max_{t\in R} x_1(t) = \max_{t\in[0,T]} x_1(t)$. Then $x_1'(t_0) = 0$, which together with the first equation of (3.2) leads to $x_2(t_0) = \varphi_p(\frac{1}{\lambda}x_1(t_0)) = 0$, $\forall\lambda \in (0, 1)$. Furthermore, we can conclude

$$x_2'(t_0) \leqslant 0. \tag{3.3}$$

In fact, if $x_2'(t_0) > 0$, then there is a constant $\sigma > 0$ such that $x_2'(t) > 0$ for $t \in [t_0, t_0 + \sigma]$, and then $x_2(t) > x_2(t_0) = 0$ for $t \in [t_0, t_0 + \sigma]$. Hence, $x_1'(t) = \lambda\varphi_q(x_2(t)) > 0$ for $t \in [t_0, t_0 + \sigma]$, i.e., $x_1(t) > x_1(t_0)$, $t \in [t_0, t_0 + \sigma]$, which contradicts the assumption of $x_1(t_0) = \max_{t\in R} x_1(t)$. This proves (3.3). From the second equation of (3.2), we have $-\lambda g(x_1(t_0 - \tau(t_0))) - \lambda f(0) + \lambda e(t_0) \leqslant 0$, i.e.,

$$g\big(x_1\big(t_0 - \tau(t_0)\big)\big) \geqslant -|e|_0 - \big|f(0)\big|,$$

which together with assumption [H$_1$] implies

$$x_1\big(t_0 - \tau(t_0)\big) \leqslant D. \tag{3.4}$$

Similarly, if t_1 is the minimum point of $x_1(t)$ on R, then

$$x_1\big(t_1 - \tau(t_1)\big) \geqslant -D. \tag{3.5}$$

From (3.4) and (3.5), it is easy to prove that there is a constant $\xi \in R$ such that

$$\big|x(\xi)\big| \leqslant D. \tag{3.6}$$

In fact, from (3.4) we see $x_1(t_0 - \tau(t_0)) \in [-D, D]$, or $x_1(t_0 - \tau(t_0)) < -D$.

(1) If $x_1(t_0 - \tau(t_0)) \in [-D, D]$, we set $\xi = t_0 - \tau(t_0)$, and then $|x_1(\xi)| \leqslant D$.
(2) If $x_1(t_0 - \tau(t_0)) < -D$, from (3.5) and the fact that the $x_1(t)$ is continuous on R, there is a point ξ between $t_0 - \tau(t_0)$ and $t_1 - \tau(t_1)$ such that $|x_1(\xi)| \leqslant D$.

This proves (3.6). So

$$|x_1|_0 = \max_{t\in[0,T]}\big|x_1(t)\big| = \max_{t\in[\xi,\xi+T]}\big|x_1(t)\big| \leqslant \big|x_1(\xi)\big| + \int_\xi^{\xi+T} \big|x_1'(s)\big|\,ds \leqslant D + \int_0^T \big|x_1'(s)\big|\,ds. \tag{3.7}$$

On the other hand, by substituting $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$ into the second equation of (3.2), we get

$$\left[\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right]' + \lambda f\left(\frac{1}{\lambda}x_1'(t)\right) + \lambda g\big(x_1\big(t - \tau(t)\big)\big) = \lambda e(t). \tag{3.8}$$

Multiplying both sides of Eq. (3.8) by $x_1(t)$ and integrating them on the interval $[0, T]$, we have

$$\int_0^T \left[\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right]' x_1(t) + \lambda\int_0^T g\big(x_1\big(t - \tau(t)\big)\big)x_1(t)\,dt + \lambda\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1(t)\,dt$$
$$= \lambda\int_0^T x_1(t)e(t)\,dt.$$

So

$$
\begin{aligned}
&-\frac{1}{\lambda^{p-1}}\int_0^T \bigl|x_1'(t)\bigr|^p\,dt+\lambda\int_0^T g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)x_1\bigl(t-\tau(t)\bigr)\,dt\\
&\quad+\lambda\int_0^T g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)\bigl[x_1(t)-x_1\bigl(t-\tau(t)\bigr)\bigr]\,dt+\lambda\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1(t)\,dt\\
&=\lambda\int_0^T x_1(t)e(t)\,dt,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\int_0^T \bigl|x_1'(t)\bigr|^p\,dt&=\lambda^p\int_0^T g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)x_1\bigl(t-\tau(t)\bigr)\,dt\\
&\quad+\lambda^p\int_0^T g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)\bigl[x_1(t)-x_1\bigl(t-\tau(t)\bigr)\bigr]\,dt\\
&\quad+\lambda^p\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1(t)\,dt-\lambda^p\int_0^T x_1(t)e(t)\,dt.
\end{aligned}
\tag{3.9}
$$

From assumption $(2r+\alpha)\delta^{(p-1)/p}T^{1/p}+\alpha T<1$, we see that there is a constant $\varepsilon_0>0$, such that $[2(r+\varepsilon_0)+\alpha]\delta^{(p-1)/p}T^{1/p}+\alpha T<1$. For such a constant $\varepsilon_0>0$, it follows from assumption $[\mathrm{H}_2]$ that there is a constant $\rho>D$ such that

$$
\bigl|g(u)\bigr|<(r+\varepsilon_0)|u|^{p-1},\quad \text{for } u\in R \text{ with } u<-\rho. \tag{3.10}
$$

From assumption $[\mathrm{H}_1]$, we see

$$
ug(u)<0,\quad \text{for } u\in R \text{ with } |u|>\rho. \tag{3.11}
$$

Let $g_\rho=\max_{|u|\leqslant\rho}|g(u)|$, $E_1=\{t\in[0,T]\colon\ x_1(t-\tau(t))<-\rho\}$, $E_2=\{t\in[0,T]\colon\ |x_1(t-\tau(t))|\leqslant\rho\}$ and $E_3=\{t\in[0,T]\colon\ x_1(t-\tau(t))>\rho\}$. Then $E_1\cup E_2\cup E_3=[0,T]$. It follows from (3.9)–(3.11) that

$$
\begin{aligned}
&\int_0^T \bigl|x_1'(t)\bigr|^p\,dt\\
&\quad=\lambda^p\biggl[\int_{E_1}+\int_{E_2}+\int_{E_3}\biggr]g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)x_1\bigl(t-\tau(t)\bigr)\,dt\\
&\qquad+\lambda^p\int_0^T g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)\bigl[x_1(t)-x_1\bigl(t-\tau(t)\bigr)\bigr]\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^p\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1(t)\,dt-\lambda^p\int_0^T x_1(t)e(t)\,dt\\
&\leqslant \lambda^p\int_{E_2} g\big(x_1\big(t-\tau(t)\big)\big)x_1\big(t-\tau(t)\big)\,dt\\
&\quad+\lambda^p\int_0^T g\big(x_1\big(t-\tau(t)\big)\big)\big[x_1(t)-x_1\big(t-\tau(t)\big)\big]\,dt\\
&\quad+\lambda^p\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1(t)\,dt-\lambda^p\int_0^T x_1(t)e(t)\,dt\\
&\leqslant T\rho g_\rho+\int_0^T\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt\max_{t\in[0,T]}\big|x_1(t)-x_1\big(t-\tau(t)\big)\big|\\
&\quad+\lambda^p|x_1|_0\int_0^T\left|f\left(\frac{1}{\lambda}x_1'(t)\right)\right|dt+|x_1|_0\int_0^T\big|e(t)\big|\,dt.
\end{aligned}
\tag{3.12}
$$

Let $\Delta_1=\{t\in[0,T]\colon |\frac{1}{\lambda}x_1'(t)|\leqslant D\}$, $\Delta_2=\{t\in[0,T]\colon |\frac{1}{\lambda}x_1'(t)|>D\}$, and integrating the two sides of (3.8) on $[0,T]$, we get

$$
\int_0^T g\big(x_1\big(t-\tau(t)\big)\big)\,dt=-\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)dt,
$$

and then

$$
\begin{aligned}
\int_{E_3}\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt&=\left|\int_{E_3} g\big(x_1\big(t-\tau(t)\big)\big)\,dt\right|\\
&\leqslant\left(\int_{E_2}+\int_{E_1}\right)\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt+\left(\int_{\Delta_1}+\int_{\Delta_2}\right)\left|f\left(\frac{1}{\lambda}x_1'(t)\right)\right|dt.
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^T\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt&=\left(\int_{E_1}+\int_{E_2}+\int_{E_3}\right)\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt\\
&\leqslant 2\left(\int_{E_1}+\int_{E_2}\right)\big|g\big(x_1\big(t-\tau(t)\big)\big)\big|\,dt+\left(\int_{\Delta_1}+\int_{\Delta_2}\right)\left|f\left(\frac{1}{\lambda}x_1'(t)\right)\right|dt\\
&\leqslant 2Tg_\rho+2(r+\varepsilon)|x_1|_0^{p-1}+f_\rho T+\frac{\alpha}{\lambda^{p-1}}\int_0^T\big|x_1'(t)\big|^{p-1}\,dt+\beta T,
\end{aligned}
\tag{3.13}
$$

where $f_\rho=\max_{|u|\leqslant\rho}|f(u)|$.

Furthermore

$$
\begin{aligned}
\max_{t\in[0,T]}\left|x_1(t)-x_1\bigl(t-\tau(t)\bigr)\right| &= \max_{t\in[0,T]}\left|\int_{t-\tau(t)}^{t} x_1'(s)\,ds\right| \leqslant \max_{t\in[0,T]}\int_{t-\tau(t)}^{t}\left|x_1'(s)\right|ds\\
&\leqslant \max_{t\in[0,T]}\left|\tau(t)\right|^{(p-1)/p}\max_{t\in[0,T]}\left(\int_{t-\tau(t)}^{t}\left|x_1'(s)\right|^p ds\right)^{1/p}\\
&\leqslant \delta^{(p-1)/p}\max_{t\in[0,T]}\left(\int_{t-T}^{t}\left|x_1'(s)\right|^p ds\right)^{1/p}\\
&= \delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}.
\end{aligned}
\tag{3.14}
$$

Substituting (3.13) and (3.14) into (3.12), we get

$$
\begin{aligned}
&\int_0^T\left|x_1'(t)\right|^p dt\\
&\quad\leqslant T\rho g_\rho+\lambda^p\left[2Tg_\rho+2(r+\varepsilon_0)|x_1|_0^{p-1}+f_\rho T+\frac{\alpha}{\lambda^{p-1}}\int_0^T\left|x_1'(s)\right|^{p-1}ds+\beta T\right]\\
&\qquad\times\delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}+\lambda^p|x_1|_0\left[\frac{\alpha}{\lambda^{p-1}}\int_0^T\left|x_1'(s)\right|^{p-1}ds+\beta T\right]\\
&\qquad+|x_1|_0\int_0^T\left|e(s)\right|ds\\
&\quad\leqslant T\rho g_\rho+(2Tg_\rho+f_\rho T+\beta T)\delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}\\
&\qquad+2(r+\varepsilon_0)|x_1|_0^{p-1}\delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}+\alpha\delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}\\
&\qquad\times T^{1/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{(p-1)/p}+\alpha|x_1|_0\int_0^T\left|x'(s)\right|^{p-1}ds\\
&\qquad+\left(\beta T+\int_0^T\left|e(s)\right|ds\right)|x_1|_0.
\end{aligned}
\tag{3.15}
$$

Since

$$\begin{aligned}
&|x_1|_0 \int_0^T \left|x_1'(s)\right|^{p-1} ds \\
&\leqslant \left(D + \int_0^T \left|x_1'(t)\right| dt\right) \int_0^T \left|x_1'(s)\right|^{p-1} ds \\
&\leqslant D \int_0^T \left|x_1'(s)\right|^{p-1} ds + \int_0^T \left|x_1'(t)\right| dt \int_0^T \left|x_1'(s)\right|^{p-1} ds \\
&\leqslant DT^{1/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} \\
&\quad + T^{(p-1)/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \cdot T^{1/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} \\
&= DT^{1/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} + T \int_0^T \left|x_1'(s)\right|^p ds,
\end{aligned}$$

it follows from (3.15) and (3.7) that

$$\begin{aligned}
&\int_0^T \left|x_1'(t)\right|^p dt \\
&\leqslant T\rho g_\rho + (2Tg_\rho + f_\rho T + \beta T)\delta^{(p-1)/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad + 2(r + \varepsilon_0)|x_1|_0^{p-1} \delta^{(p-1)/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad + \left[T\alpha + \alpha\delta^{(p-1)/p} T^{1/p}\right] \int_0^T \left|x_1'(s)\right|^p ds \\
&\quad + D\alpha T^{1/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} + \left(\beta T + \int_0^T \left|e(s)\right| ds\right)|x_1|_0 \\
&\leqslant T\rho g_\rho + (2Tg_\rho + f_\rho T + \beta T)\delta^{(p-1)/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad + 2(r + \varepsilon_0)\delta^{(p-1)/p} |x_1|_0^{p-1} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p}
\end{aligned}$$

$$+\left[T\alpha+\alpha\delta^{(p-1)/p}T^{1/p}\right]\int_0^T\left|x_1'(s)\right|^p ds+D\alpha T^{1/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{(p-1)/p}$$
$$+\left[\beta T+\int_0^T\left|e(s)\right|ds\right]T^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}$$
$$+\left(\beta T+\int_0^T\left|e(s)\right|ds\right)D. \tag{3.16}$$

We will prove next that there must be a constant $M_1>0$ independent of λ such that

$$\int_0^T\left|x_1'(s)\right|^p ds\leqslant M_1. \tag{3.17}$$

Case 1. If $1<p\leqslant 2$, i.e., $0<p-1\leqslant 1$, then from (3.7)

$$|x_1|_0^{p-1}\leqslant d+\left(\int_0^T\left|x_1'(s)\right|ds\right)^{p-1},$$

where $d=D^{p-1}$. By using Hölder's inequality, we get

$$|x_1|_0^{p-1}\leqslant d+T^{1/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{(p-1)/p}. \tag{3.18}$$

Substituting (3.18) into (3.16), we get

$$\int_0^T\left|x_1'(t)\right|^p dt$$
$$\leqslant T\rho g_\rho+\left[2Tg_\rho+f_\rho T+\beta T+2(r+\varepsilon_0)d\right]\delta^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}$$
$$+\left[2(r+\varepsilon_0)T^{1/p}\delta^{(p-1)/p}+\alpha T+\alpha\delta^{(p-1)/p}T^{1/p}\right]\int_0^T\left|x_1'(s)\right|^p ds$$
$$+D\alpha T^{1/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{(p-1)/p}$$
$$+\left[\beta T+\int_0^T\left|e(s)\right|ds\right]T^{(p-1)/p}\left(\int_0^T\left|x_1'(s)\right|^p ds\right)^{1/p}+\left(\beta T+\int_0^T\left|e(s)\right|ds\right)D.$$

It follows from $[2(r+\varepsilon_0)+\alpha]\delta^{(p-1)/p}T^{1/p}+\alpha T<1$, $(p-1)/p<1$ and $1/p<1$ that (3.17) holds.

Case 2. If $p \in (2, +\infty)$, then $p - 1 > 1$. By classical elementary inequalities, we see that there is a constant $h(p) > 0$ which is dependent on p only, such that

$$(1+x)^p < 1 + (1+p)x, \quad \forall x \in \big(0, h(p)\big]. \tag{3.19}$$

Since $e(t)$ is not a constant, it is obvious that $x_1(t)$ is not a constant, i.e., $\int_0^T |x_1'(s)|\,ds > 0$. So

$$\begin{aligned}
&|x_1|_0^{p-1}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad\leqslant \left(D + \int_0^T \left|x_1'(s)\right| ds\right)^{p-1} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad= \left(\int_0^T \left|x_1'(s)\right| ds\right)^{p-1} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \left(1 + \frac{D}{\int_0^T |x_1'(s)|\,ds}\right)^{p-1}.
\end{aligned} \tag{3.20}$$

Without loss of generality, we suppose $\int_0^T |x_1'(s)|\,ds > D/h$, then $\frac{D}{\int_0^T |x_1'(s)|\,ds} < h$. By (3.19) we see

$$\left(1 + \frac{D}{\int_0^T |x_1'(s)|\,ds}\right)^{p-1} < 1 + \frac{pD}{\int_0^T |x_1'(s)|\,ds},$$

and then by (3.20), we have

$$\begin{aligned}
&|x_1|_0^{p-1}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad\leqslant \left(\int_0^T \left|x_1'(s)\right| ds\right)^{p-1} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\qquad + pD\left(\int_0^T \left|x_1'(s)\right| ds\right)^{p-2} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad\leqslant T^{1/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\qquad + pD\left[T^{(p-1)/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p}\right]^{p-2} \left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\quad= T^{1/p}\int_0^T \left|x_1'(s)\right|^p ds + pDT^{(p-1)(p-2)/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p}.
\end{aligned}$$

Substituting the above formula into (3.16), we get

$$
\begin{aligned}
&\int_0^T \left|x_1'(t)\right|^p dt \\
&\quad \leqslant T\rho g_\rho + (2Tg_\rho + f_\rho T + \beta T)\delta^{(p-1)/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} \\
&\qquad + \left[2(r+\varepsilon_0)T^{1/p}\delta^{(p-1)/p} + \alpha T + \alpha\delta^{(p-1)/p}T^{1/p}\right]\int_0^T \left|x_1'(s)\right|^p ds \\
&\qquad + 2pD(r+\varepsilon_0)\delta^{(p-1)/p}T^{(p-1)(p-2)/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} \\
&\qquad + D\alpha T^{1/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{(p-1)/p} \\
&\qquad + \left[\beta T + \int_0^T \left|e(s)\right| ds\right]T^{(p-1)/p}\left(\int_0^T \left|x_1'(s)\right|^p ds\right)^{1/p} + \left(\beta T + \int_0^T \left|e(s)\right| ds\right)D.
\end{aligned}
\tag{3.21}
$$

Since $2(r+\varepsilon_0)T^{1/p}\delta^{(p-1)/p} + \alpha T + \alpha\delta^{(p-1)/p}T^{1/p} < 1$, $1/p < 1$ and $(p-1)/p < 1$, it follows from (3.21) that (3.17) is also satisfied. This proves (3.17). Hence

$$
|x_1|_0 \leqslant D + \int_0^T \left|x_1'(s)\right| ds < D + T^{(p-1)/p}(M_1)^{1/p} := M_0. \tag{3.22}
$$

Again from the first equation of (3.2), we have

$$
\int_0^T \left|x_2(s)\right|^{q-2} x_2(s)\, ds = 0,
$$

which implies that there is a constant $t_2 \in [0, T]$ such that $x_2(t_2) = 0$. So

$$
|x_2|_0 \leqslant \int_0^T \left|x_2'(s)\right| ds \leqslant \left(\int_0^T \left|x_2'(t)\right|^2 dt\right)^{1/2} T^{1/2}. \tag{3.23}
$$

Multiplying both sides of the second equation of (3.2) by $x_2'(t)$ and integrating them on the interval $[0, T]$, we obtain

$$
\begin{aligned}
&\int_0^T \left|x_2'(s)\right|^2 ds \\
&\quad = -\lambda \int_0^T f\big(\varphi_q\big(x_2(t)\big)\big)x_2'(t)\, dt - \lambda \int_0^T g\big(x_1\big(t-\tau(t)\big)\big)x_2'(t)\, dt + \lambda \int_0^T e(t)x_2'(t)\, dt
\end{aligned}
$$

$$= -\lambda \int_0^T g\big(x_1\big(t-\tau(t)\big)\big)x_2'(t)\,dt + \lambda \int_0^T e(t)x_2'(t)\,dt$$

$$\leqslant \int_0^T \big|g\big(x_1\big(t-\tau(t)\big)\big)x_2'(t)\big|\,dt + \int_0^T \big|e(t)x_2'(t)\big|\,dt$$

$$\leqslant \big[g_{M_0} + |e|_0\big]T^{1/2}\left(\int_0^T \big|x_2'(t)\big|^2\,dt\right)^{1/2},$$

where $g_{M_0} = \max_{|u|\leqslant M_0} |g(u)|$. So

$$\left(\int_0^T \big|x_2'(t)\big|^2\,dt\right)^{1/2} \leqslant \big[g_{M_0} + |e|_0\big]T^{1/2},$$

and then by using (3.23) we get

$$|x_2|_0 \leqslant \big[g_{M_0} + |e|_0\big]T := M_1.$$

Let $\Omega_2 = \{x\colon\ x \in \operatorname{Ker} L,\ QNx = 0\}$. If $x \in \Omega_2$, then $x \in R^2$ is a constant vector with

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ f\big(|x_2|^{q-2}x_2\big) + g(x_1) = \bar{e}. \end{cases}$$

So $x_2 = 0$ and by assumption [H_1], we see $|x_1| \leqslant D$, which implies $\Omega_2 \subset \Omega_1$.

Now, if we set $\Omega = \{x\colon\ x = (x_1, x_2)^\top \in X,\ |x_1|_0 < M_0 + 1,\ |x_2|_0 < M_1 + 1\}$, then $\Omega \supset \Omega_1 \cup \Omega_1$. So conditions (1) and (2) of Lemma 2.1 are satisfied. The remainder is to verify condition (3) of Lemma 2.1. In order to do it, let

$$J : \operatorname{Im} Q \to \operatorname{Ker} L, \qquad J(x_1, x_2) = (x_1, x_2),$$

$\Delta_\varepsilon = \{x\colon\ x = (x_1, x_2)^\top \in R^2\colon\ |x_1| < M_0,\ |x_2| < \varepsilon\}$. It is easy to see that for arbitrary small $\varepsilon > 0$, equation $QN(x) = (0, 0)^\top$, i.e.,

$$\begin{cases} \varphi_q(x_2) = 0, \\ -f\big(\varphi_q(x_2)\big) - g(x_1) + \bar{e} = 0, \end{cases}$$

has no solution in $\overline{(\Omega \cap \operatorname{Ker} L)\backslash \Delta_\varepsilon}$. So

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{JQN, \Delta_\varepsilon, 0\}.$$

Let

$$QN_0 = \begin{pmatrix} 0 \\ -f(0) - g(x_1) + \bar{e} \end{pmatrix}.$$

If $x \in \partial \Delta_\varepsilon$,

$$\big\|JQN(x) - JQN_0(x)\big\| \leqslant \max_{|x_2|\leqslant\varepsilon} \big\{\big|f\big(\varphi_q(x_2)\big) - f(0)\big| + \big|\varphi_q(x_2)\big|\big\},$$

which implies $\|JQN(x) - JQN_0(x)\| \to 0$, as $\varepsilon \to 0$. So if $\varepsilon > 0$ is sufficiently small,

$$\deg\{JQN, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_\varepsilon, 0\}.$$

In view of $\dim QN_0 = 1$, it follows that

$$\deg\{JQN_0, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_0, 0\},$$

where $\Delta_0 = \{x\colon\ x \in R,\ |x| < M_0\} \subset R$. By using assumption [$H_1$], we see $\deg\{JQN_0, \Delta_0, 0\} \neq 0$, i.e.,

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{JQN_0, \Delta_0, 0\} \neq 0.$$

Therefore, by using Lemma 2.1, we see that equation

$$Lx = Nx$$

has a solution $x^*(t) = (x_1^*(t), x_2^*(t))^\top$ on $\overline{\Omega}$, i.e., Eq. (1.6) has a T-periodic solution $x_1^*(t)$ with $|x_1^*|_0 \leqslant M_0 + 1$. Clearly $x_1^*(t)$ is not a constant. Otherwise, by substituting $x_1^*(t) \equiv C$ (constant) into Eq. (1.6), we have $f(0) + g(C) = e(t)$, which contradicts the assumption that $e(t)$ is not a constant. $\square$

Remark 3.1. If we replace assumption [H_2] by [H_2']: $\lim_{x\to+\infty} \frac{|g(x)|}{|x|^{p-1}} \leqslant r \in [0, +\infty)$, then the result of Theorem 3.1 is also satisfied. The only difference of the proof is that formula (3.10) should be modified by

$$\bigl|g(u)\bigr| < (r + \varepsilon_0)|u|^{p-1}, \quad \text{for } u \in R \text{ with } u > \rho,$$

and the inequality

$$\begin{aligned}
&\biggl(\int\limits_{E_1} + \int\limits_{E_2} + \int\limits_{E_3}\biggr)\bigl|g\bigl(x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr|\,dt \\
&\quad\leqslant 2\biggl(\int\limits_{E_1} + \int\limits_{E_2}\biggr)\bigl|g\bigl(x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr|\,dt + \biggl(\int\limits_{\Delta_1} + \int\limits_{\Delta_2}\biggr)\biggl|f\biggl(\frac{1}{\lambda}x_1'(t)\biggr)\biggr|\,dt
\end{aligned}$$

in (3.13) should be replaced by

$$\begin{aligned}
&\biggl(\int\limits_{E_1} + \int\limits_{E_2} + \int\limits_{E_3}\biggr)\bigl|g\bigl(x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr|\,dt \\
&\quad\leqslant 2\biggl(\int\limits_{E_3} + \int\limits_{E_2}\biggr)\bigl|g\bigl(x_1\bigl(t - \tau(t)\bigr)\bigr)\bigr|\,dt + \biggl(\int\limits_{\Delta_1} + \int\limits_{\Delta_2}\biggr)\biggl|f\biggl(\frac{1}{\lambda}x_1'(t)\biggr)\biggr|\,dt.
\end{aligned}$$

Example 3.1 considers the following equation:

$$x''(t) + \frac{1}{4\pi}x'(t)\sin x'(t) + g\bigl(x\bigl(t - \theta|\sin t|\bigr)\bigr) = \cos t, \tag{3.24}$$

where

$$g(u) = \begin{cases} -ue^u, & u \geqslant 0; \\ \frac{1}{2}u, & u < 0; \end{cases}$$

$\theta \in (0, 1)$ is a parameter. Corresponding to Eq. (1.6), we have $p = 2$, $T = 2\pi$, $\delta = \theta$. Since

$$\lim_{x\to-\infty}\frac{|g(x)|}{|x|^{p-1}}=\lim_{x\to-\infty}\frac{u/2}{u}=1/2$$

and from Eq. (3.24), we can choose $r=1/2$, $\alpha=\frac{1}{4\pi}$ and $D=2$ such that assumptions [H_1]–[H_3] are satisfied. So if $\theta\in(0,\frac{2\pi}{(4\pi+1)^2})$, then

$$(2r+\alpha)\delta^{(p-1)/p}T^{1/p}+\alpha T<1.$$

Therefore, by using Theorem 3.1, Eq. (3.24) has at least one non-constant 2π-periodic solution for all $\theta\in(0,2\pi(4\pi+1)^{-2})$.

Remark 3.2. Even if $p=2$, the result of Example 3.1 cannot be obtained by [1–3]. The reason for this is that

$$g(u)=\begin{cases}-ue^u, & u\geqslant 0;\\ \frac{1}{2}u, & u<0;\end{cases}$$

does not satisfy global linear growth condition (1.5), and $f(u)=\frac{1}{4\pi}u\sin u$ does not satisfy conditions (1.2)–(1.4) either.

The aim of the following theorem is to remove condition [H_3] imposed on $f(x)$ in Theorem 3.1.

Theorem 3.2. *Suppose* $p\geqslant 2$, $\delta:=\max_{t\in[0,T]}\tau(t)\in[0,T]$, $e(t)$ *is not a constant, and assumptions* [H_1] *and* [H_4] *hold. Then Eq.* (1.6) *has at least one non-constant T-periodic solution, if one of the following conditions holds*:

[A_1] $p=2$ *and* $\frac{l\delta T}{2\pi}<1$.
[A_2] $p>2$.

Proof. Let $x(t)=(x_1(t),x_2(t))^\top$ be an arbitrary T-periodic solution to Eqs. (3.2). By assumption [H_1] and from the proof of Theorem 3.1, one can find

$$|x_1|_0\leqslant D+\int_0^T\left|x_1'(s)\right|ds. \tag{3.25}$$

If set the substitution $y(t)=\varphi_p(\frac{1}{\lambda}x_1'(t))$, then

$$\int_0^T\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'f\left(\frac{1}{\lambda}x_1'(t)\right)dt=\int_0^T f\left(\varphi_q\left(y(t)\right)\right)y'(t)\,dt=0.$$

So multiplying both sides of Eq. (3.8) by $(\varphi_p(\frac{1}{\lambda}x_1'(t)))'$ and integrating them on the interval $[0,T]$, we get

$$\int_0^T\left[\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right]^2dt$$

$$
\begin{aligned}
&= -\lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' g\big(x_1\big(t-\tau(t)\big)\big)\,dt + \lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' e(t)\,dt \\
&= -\lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' g\big(x_1(t)\big)\,dt \\
&\quad - \lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' \big[g\big(x_1\big(t-\tau(t)\big)\big) - g\big(x_1(t)\big)\big]\,dt \\
&\quad + \lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' e(t)\,dt.
\end{aligned} \tag{3.26}
$$

From [H$_4$], we see

$$
-\lambda \int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' g\big(x_1(t)\big)\,dt = \lambda^p \int_0^T \big|x_1'(t)\big|^{p-2}\big(x_1'(t)\big)^2 g'\big(x_1(t)\big)\,dt \leqslant 0 \tag{3.27}
$$

and

$$
\begin{aligned}
&\int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' \big[g\big(x_1\big(t-\tau(t)\big)\big) - g\big(x_1(t)\big)\big]\,dt \\
&\quad \leqslant \int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right| \big|g\big(x_1\big(t-\tau(t)\big)\big) - g\big(x_1(t)\big)\big|\,dt \\
&\quad \leqslant l \int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right| \big|x_1(t) - x_1\big(t-\tau(t)\big)\big|\,dt \\
&\quad \leqslant l \left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \left(\int_0^T \big|x_1(t) - x_1\big(t-\tau(t)\big)\big|^2\,dt\right)^{1/2}.
\end{aligned}
$$

By using Lemma 2.2, we have

$$
\begin{aligned}
&\int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' \big[g\big(x_1\big(t-\tau(t)\big)\big) - g\big(x_1(t)\big)\big]\,dt \\
&\quad \leqslant \delta l \left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \left(\int_0^T \big|x_1'(t)\big|^2\,dt\right)^{1/2}.
\end{aligned} \tag{3.28}
$$

Moreover,

$$
\int_0^T \left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)' e(t)\,dt \leqslant \left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \left(\int_0^T \big|e(t)\big|^2\,dt\right)^{1/2}. \tag{3.29}
$$

Substituting (3.27)–(3.29) into (3.24), we have

$$\int_0^T \left[\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right]^2 dt \leqslant \delta l\left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2}\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{1/2} + \left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2},$$

which leads to

$$\left(\int_0^T \left|\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \leqslant \delta l\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{1/2} + \left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2}. \tag{3.30}$$

Considering $(\varphi_p(\frac{1}{\lambda}x_1'(t)))' = \frac{1}{\lambda^{p-1}}(\varphi_p(x_1'(t)))'$, it follows from (3.30) that

$$\left(\int_0^T \left|\left(\varphi_p\left(x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \leqslant \delta l\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{1/2} + \left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2}. \tag{3.31}$$

Case 1. If condition [A_1] holds, i.e., $p = 2$ and $\frac{\delta l T}{2\pi} < 1$, then by (3.31)

$$\left(\int_0^T \left|x_1''(s)\right|^2 ds\right)^{1/2} = \left(\int_0^T \left|\left(\varphi_p\left(x_1'(t)\right)\right)'\right|^2 dt\right)^{1/2} \leqslant \delta l\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{1/2} + \left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2}.$$

It follows from Wirtinger's inequality $\int_0^T |x_1'(s)|^2\,ds \leqslant \frac{T^2}{4\pi^2}\int_0^T |x_1''(s)|^2\,ds$ that

$$\left(\int_0^T \left|x_1'(s)\right|^2 ds\right)^{1/2} \leqslant \frac{T}{2\pi}\left(\int_0^T \left|x_1''(t)\right|^2 dt\right)^{1/2} \leqslant \frac{\delta l T}{2\pi}\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{1/2} + \frac{T}{2\pi}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2}.$$

So

$$\left(\int_0^T \left|x_1'(s)\right|^2 ds\right)^{1/2} \leqslant \frac{T}{2\pi}\left[1 - \frac{\delta l T}{2\pi}\right]^{-1}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2} := M_2.$$

Thus by (3.25)

$$|x_1|_0 \leqslant D + T^{1/2}\left(\int_0^T \left|x_1'(s)\right|^2 ds\right)^{1/2} \leqslant D + T^{1/2}M_2 := M_3.$$

Case 2. Suppose condition [A_2] holds, i.e., $p > 2$. Since $\int_0^T x_1'(s)\,ds = 0$, there must be a point $\xi \in [0, T]$ such that $x_1'(\xi) = 0$. So $\varphi_p(x_1'(\xi)) = 0$, and then

$$\begin{aligned}
\left|\varphi_p\left(x_1'\right)\right|_0 &= \max_{t\in[0,T]}\left|\varphi_p\left(x_1'(t)\right)\right| = \max_{t\in[\xi,\xi+T]}\left|\varphi_p\left(x_1'(t)\right)\right| \\
&= \max_{t\in[\xi,\xi+T]}\left|\int_\xi^t \left(\varphi_p\left(x_1'(s)\right)\right)'\,ds\right| \leqslant \int_\xi^{\xi+T} \left|\left(\varphi_p\left(x_1'(s)\right)\right)'\right|\,ds \\
&= \int_0^T \left|\left(\varphi_p\left(x_1'(s)\right)\right)'\right|\,ds \leqslant T^{1/2}\left(\int_0^T \left|\left(\varphi_p\left(x_1'(s)\right)\right)'\right|^2\,ds\right)^{1/2},
\end{aligned}$$

which together with (3.31) implies

$$\begin{aligned}
\left|x_1'\right|_0^{p-1} &= \left|\varphi_p\left(x_1'\right)\right|_0 \\
&\leqslant T^{1/2}\delta l\left(\int_0^T \left|x_1'(t)\right|^2\,dt\right)^{1/2} + T^{1/2}\left(\int_0^T \left|e(t)\right|^2\,dt\right)^{1/2} \\
&\leqslant T\delta l\left|x_1'\right|_0 + T^{1/2}\left(\int_0^T \left|e(t)\right|^2\,dt\right)^{1/2}.
\end{aligned}$$

Since $p - 1 > 1$, there must be a constant $R_0 > 0$ such that

$$\left|x_1'\right|_0^{p-1} \leqslant R_0,$$

which together with (3.25) implies

$$|x_1|_0 \leqslant D + TR_0^{1/(p-1)} := R_1.$$

Let $\overline{M} = \max\{R_1, M_3\}$. Then in either Case 1 or Case 2, we always have

$$|x_1|_0 \leqslant \overline{M}.$$

The rest of the proof of the theorem is identical to that of Theorem 3.1. $\square$

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